The Perceptron Algorithm

Machine Learning

THE UNIVERSITY OF UTAH

Some slides based on lectures from Dan Roth, Avrim Blum and others

Outline

- The Perceptron Algorithm
- Variants of Perceptron
- Perceptron Mistake Bound

Where are we?

- The Perceptron Algorithm
- Variants of Perceptron
- Perceptron Mistake Bound

Recall: Linear Classifiers

Inputs are d dimensional vectors, denoted by $\mathbf{x}$

Output is a label $y \in \{-1, 1\}$

Linear Threshold Units classify an example $\mathbf{x}$ using parameters $\mathbf{w}$ (a d dimensional vector) and b (a real number) according the following classification rule

Output = $\operatorname{sign}(\mathbf{w}^{\mathrm{T}}\mathbf{x} + b) = \operatorname{sign}(\sum_i w_i x_i + b)$

$$\mathbf{w}^{\mathrm{T}}\mathbf{x} + b \geq 0 \Rightarrow y = +1$$
$$\mathbf{w}^{\mathrm{T}}\mathbf{x} + b < 0 \Rightarrow y = -1$$

b is called the bias term

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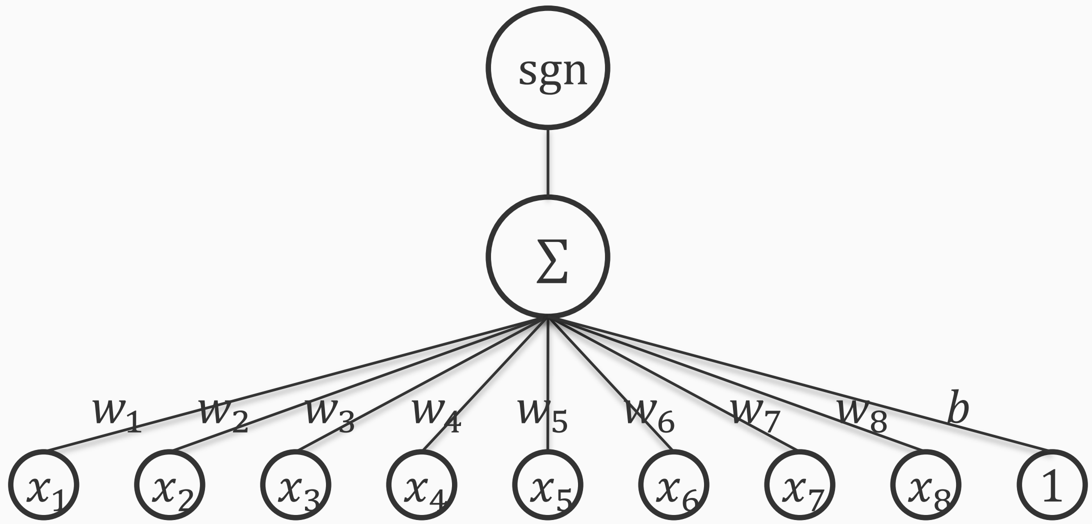
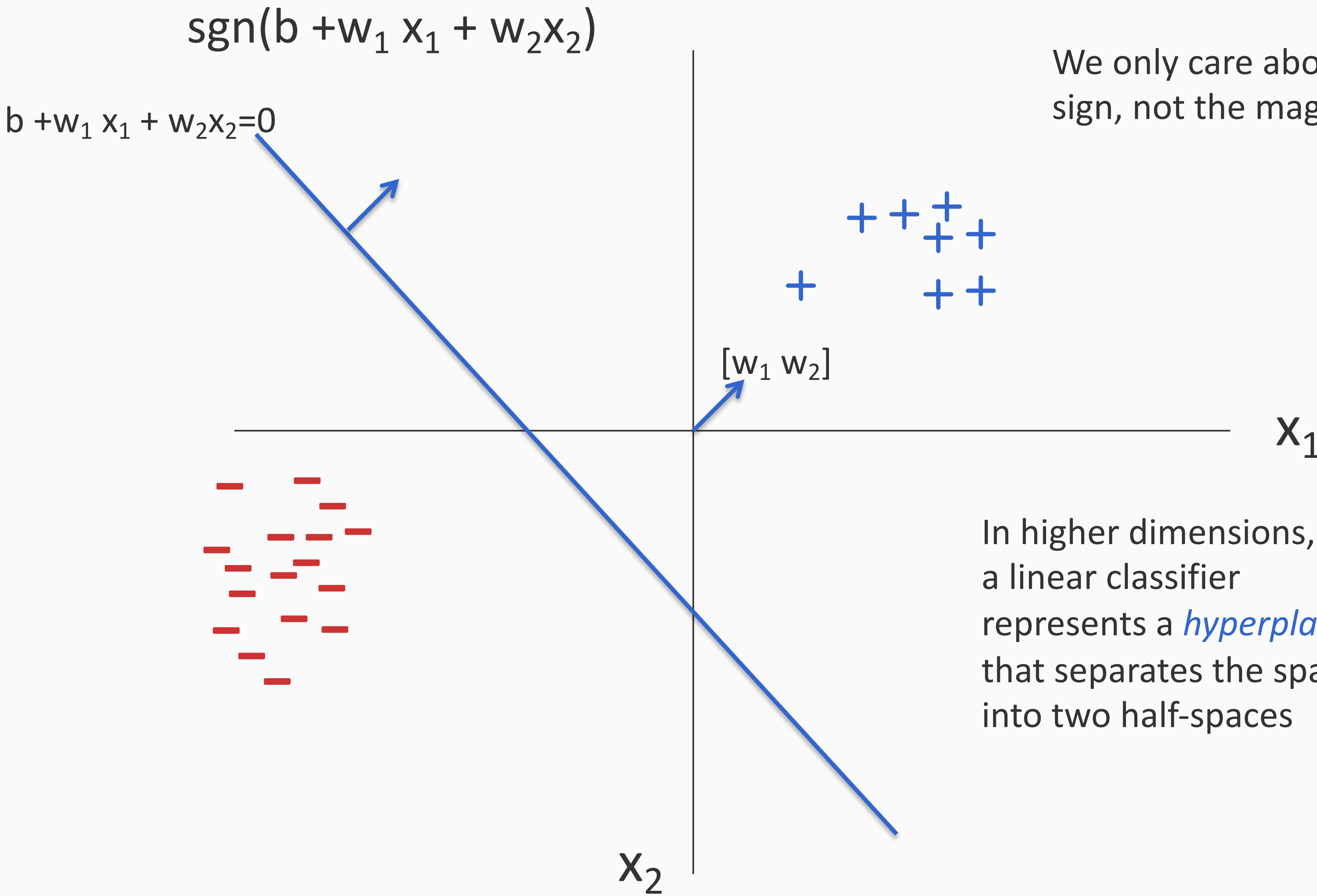


The geometry of a linear classifier



We only care about the sign, not the magnitude

In higher dimensions, a linear classifier represents a *hyperplane* that separates the space into two half-spaces

The Perceptron

REPORT NO. 85-460-1

THE PERCEPTRON

A PERCEIVING AND RECOGNIZING AUTOMATON

(PROJECT PARA)

January, 1957

Prepared by:

Frank Rosenblatt,
Project Engineer

Psychological Review
Vol. 65, No. 6, 1958

THE PERCEPTRON: A PROBABILISTIC MODEL FOR INFORMATION STORAGE AND ORGANIZATION IN THE BRAIN [1]

F. ROSENBLATT

Cornell Aeronautical Laboratory

The Perceptron algorithm

- Rosenblatt 1958
  - (Though there were some hints of a similar idea earlier, eg: Agmon 1954)
- The goal is to find a separating hyperplane
  - For separable data, guaranteed to find one
- An online algorithm
  - Processes one example at a time
- Several variants exist
  - We will see these briefly at towards the end

The Perceptron algorithm

Input: A sequence of training examples $(\mathbf{x}_1, y_1), (\mathbf{x}_2, y_2), \cdots$ where all $\mathbf{x}_i \in \Re^d, y_i \in \{-1, 1\}$

1. Initialize $\mathbf{w}_0 = 0 \in \Re^d$
2. For each training example $(\mathbf{x}_i, y_i)$:
   1. Predict $\mathrm{y}' = \mathrm{sgn}(\mathbf{w}_t^T \mathbf{x}_i)$
   2. If $\mathrm{y}' \neq y_i$:
      - Update $\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t + r(y_i \mathbf{x}_i)$
3. Return final weight vector

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Remember:
Prediction = sgn($\mathbf{w}^T\mathbf{x}$)

There is typically a bias term also ($\mathbf{w}^T\mathbf{x}$ + b), but the bias may be treated as a constant feature and folded into **w**

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Footnote: For some algorithms it is mathematically easier to represent False as -1, and at other times, as 0. For the Perceptron algorithm, treat -1 as false and +1 as true.

The Perceptron algorithm

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r is the learning rate, a small positive number less than 1

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Update only on error. A mistake-driven algorithm

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Update only on error. A mistake-driven algorithm

Mistake can be written as $\mathrm{y}_i \mathbf{w}_t^T \mathbf{x}_i \leq 0$

This is the simplest version. We will see more robust versions shortly

Intuition behind the update

Mistake on positive: $\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t + r\mathbf{x}_i$
Mistake on negative: $\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t - r\mathbf{x}_i$

Suppose we have made a mistake on a positive example

That is, $y = +1$ and $\mathbf{w}_t^{\mathrm{T}}\mathbf{x} \leq 0$

Call the new weight vector $\mathbf{w}_{t+1} = \mathbf{w}_t + \mathbf{x}$ (say r = 1)

The new dot product is $\mathbf{w}_{\mathrm{t+1}}^{\mathrm{T}}\mathbf{x} = (\mathbf{w}_t + \mathbf{x})^{\mathrm{T}}\mathbf{x} = \mathbf{w}_t^{\mathrm{T}}\mathbf{x} + \mathbf{x}^{\mathbf{T}}\mathbf{x} \geq \mathbf{w}_{\mathbf{t}}^{\mathbf{T}}\mathbf{x}$

For a positive example, the Perceptron update will increase the score assigned to the same input

Similar reasoning for negative examples

Mistake on positive: $\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t + r\mathbf{x}_i$
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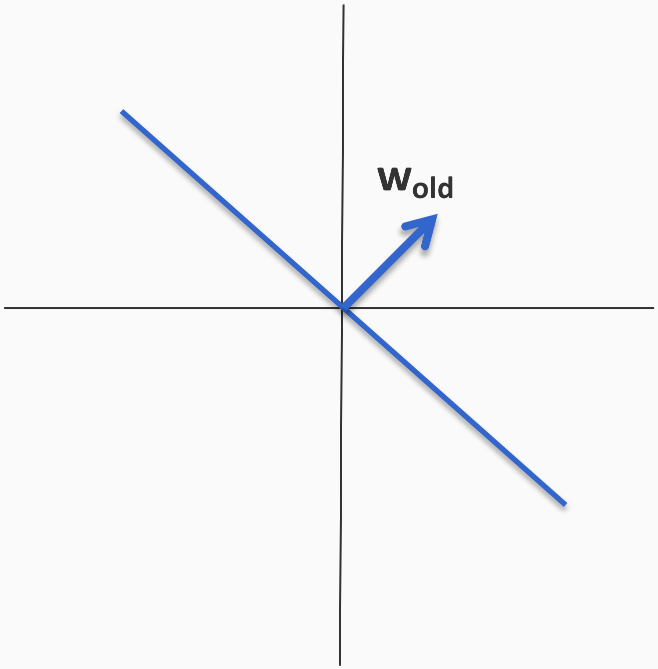
Geometry of the perceptron update

Predict



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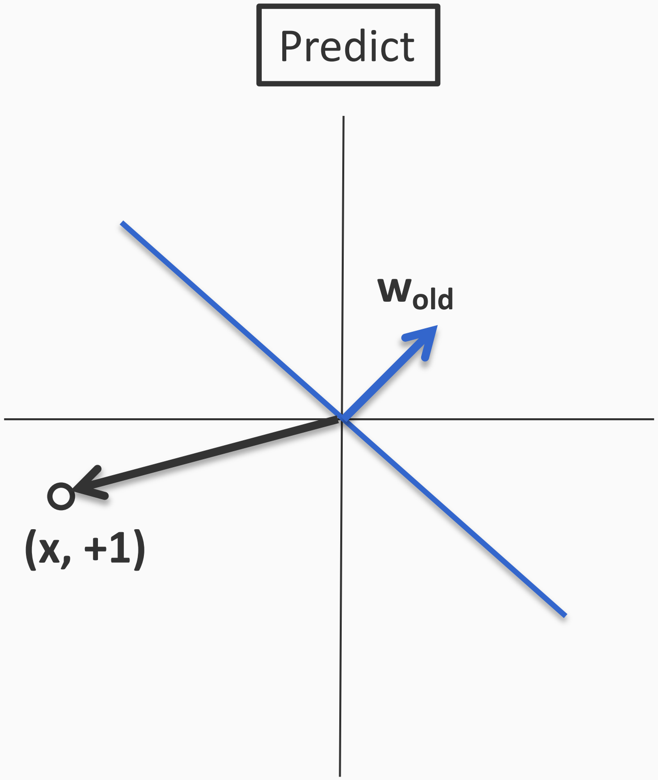

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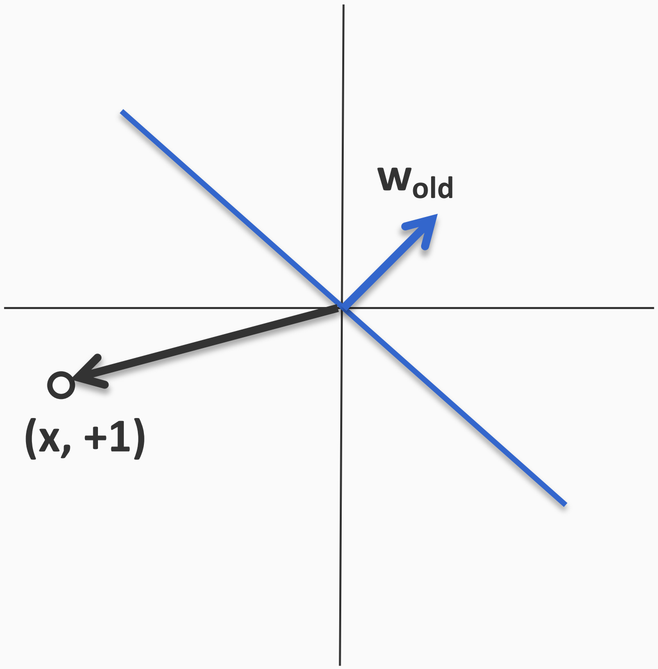

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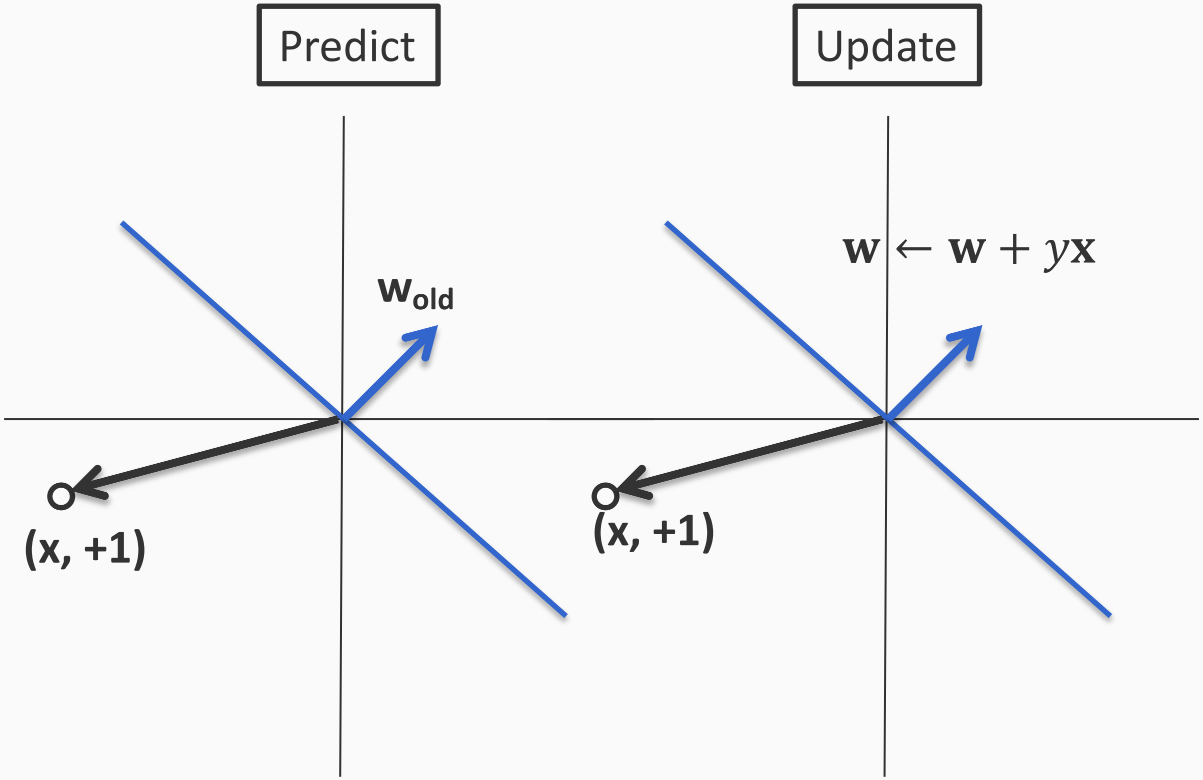

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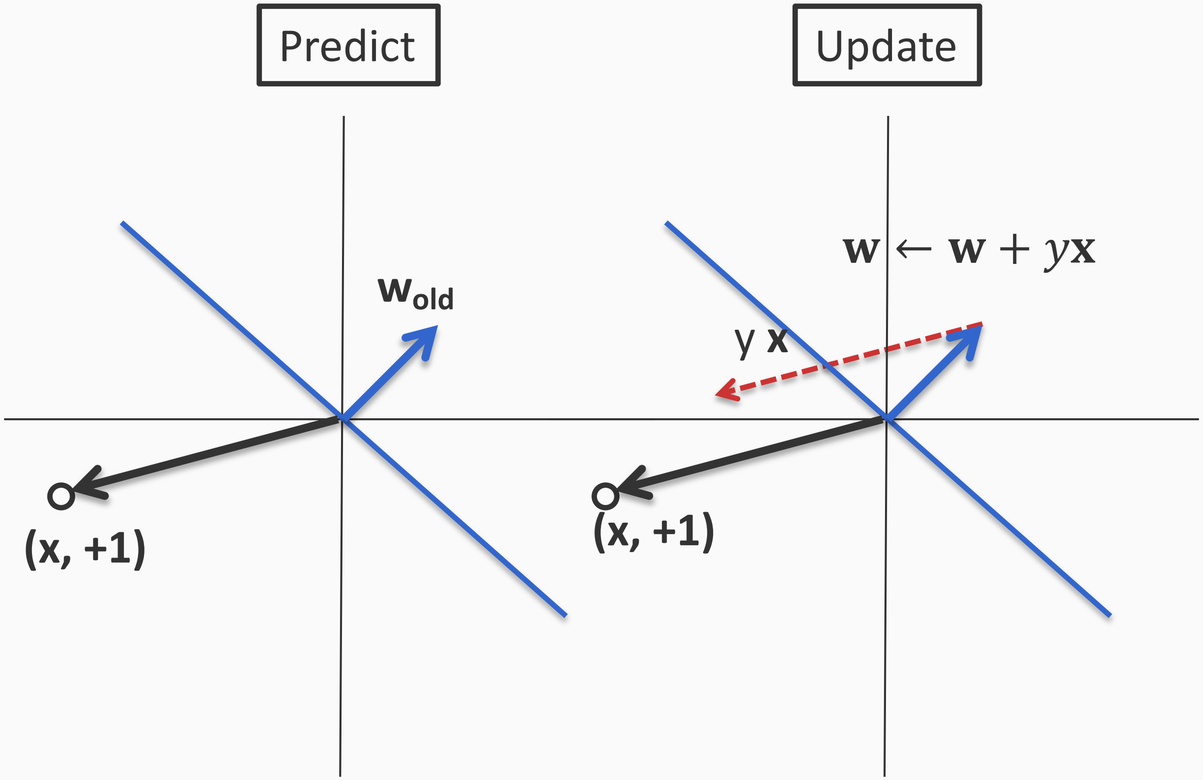

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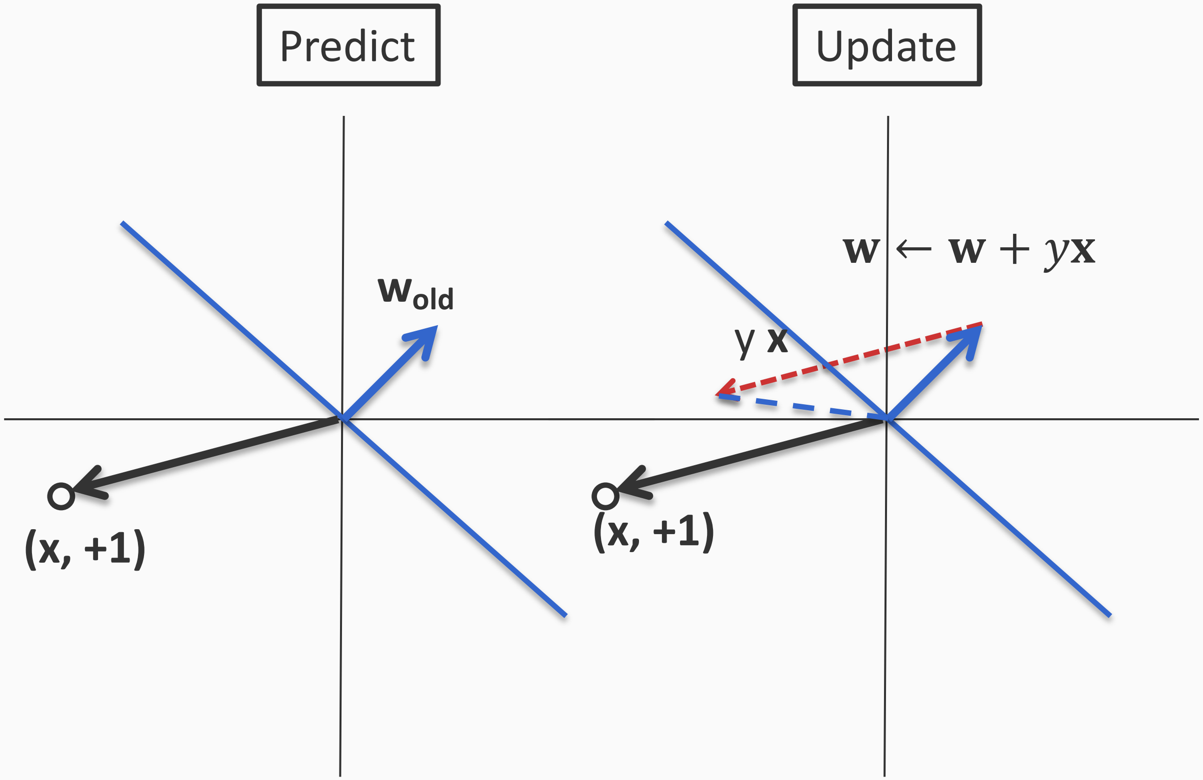

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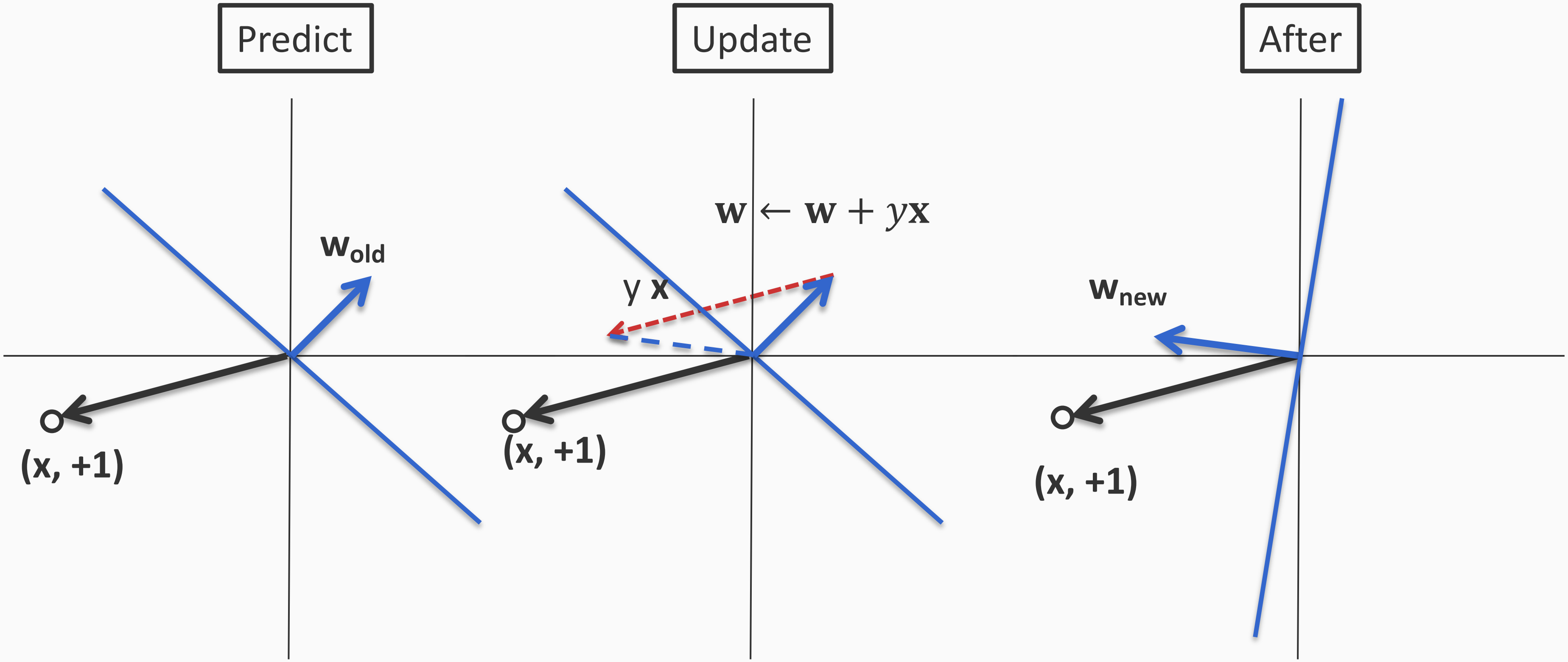

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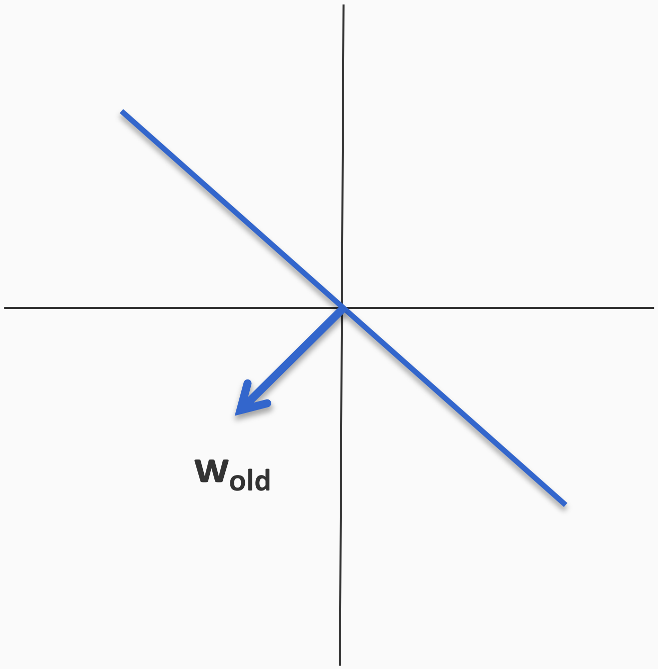

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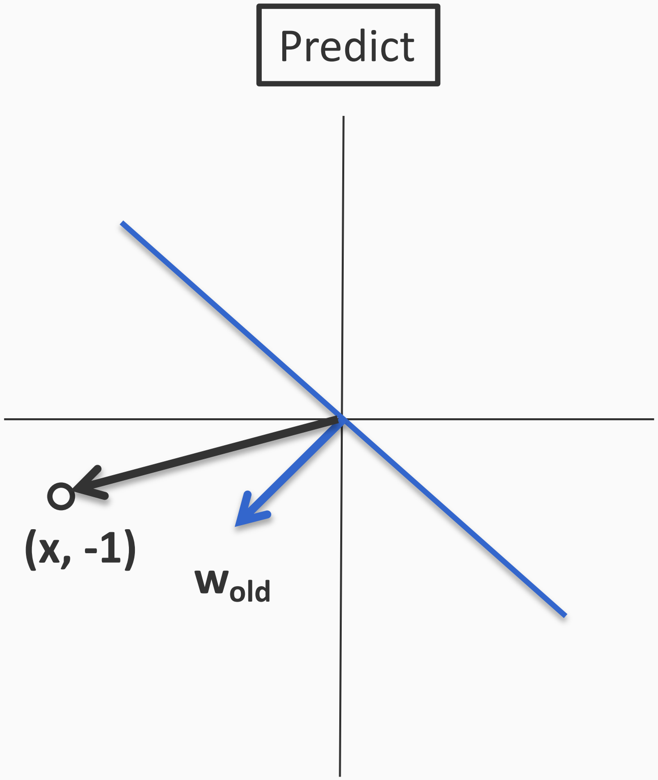

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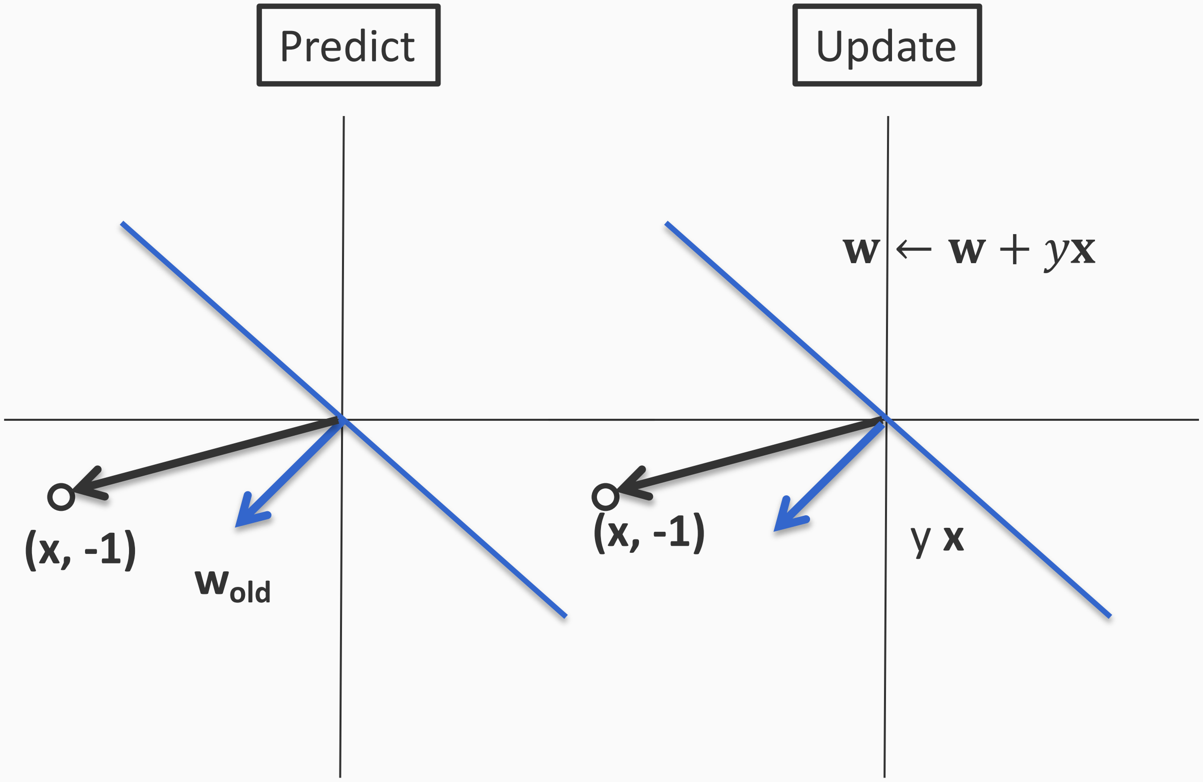

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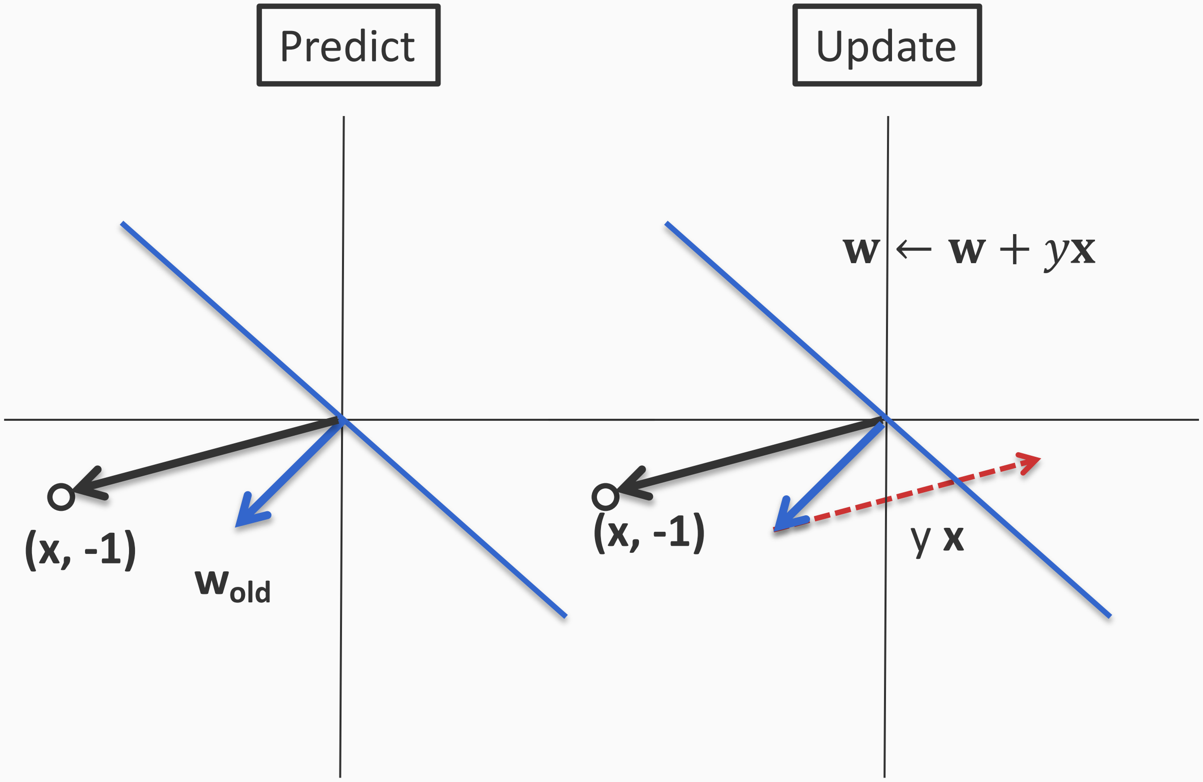

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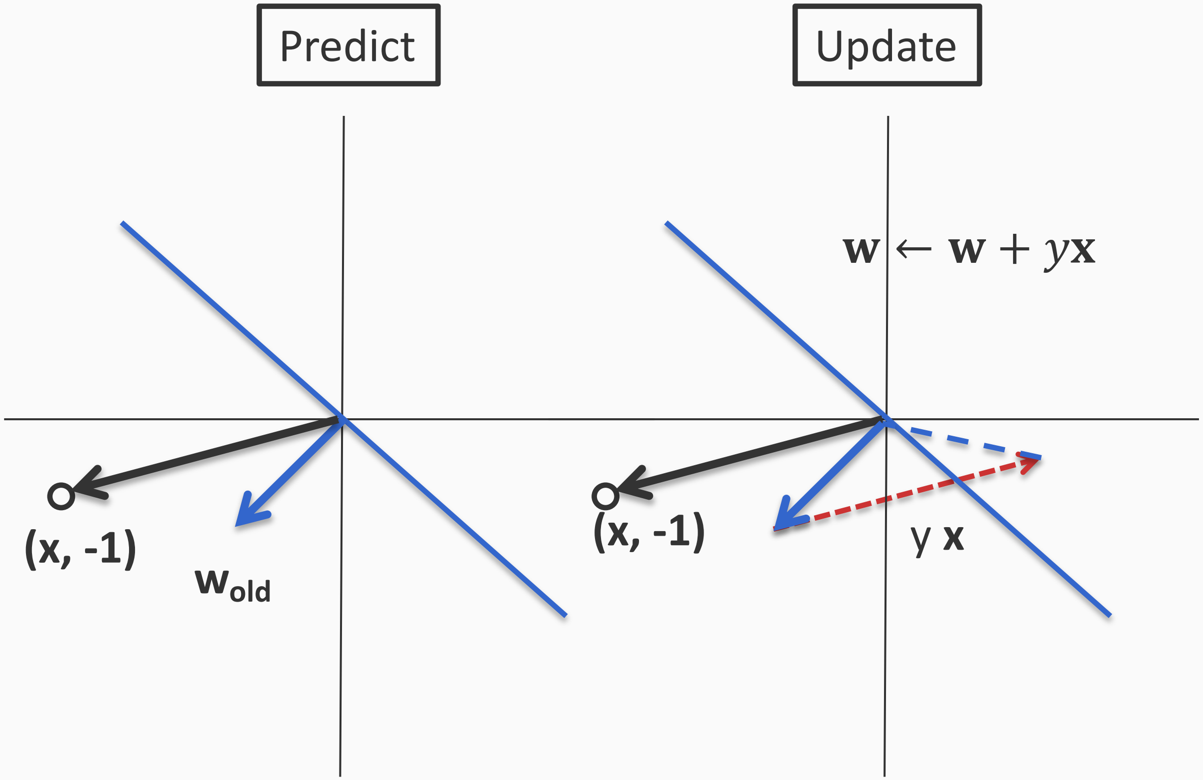

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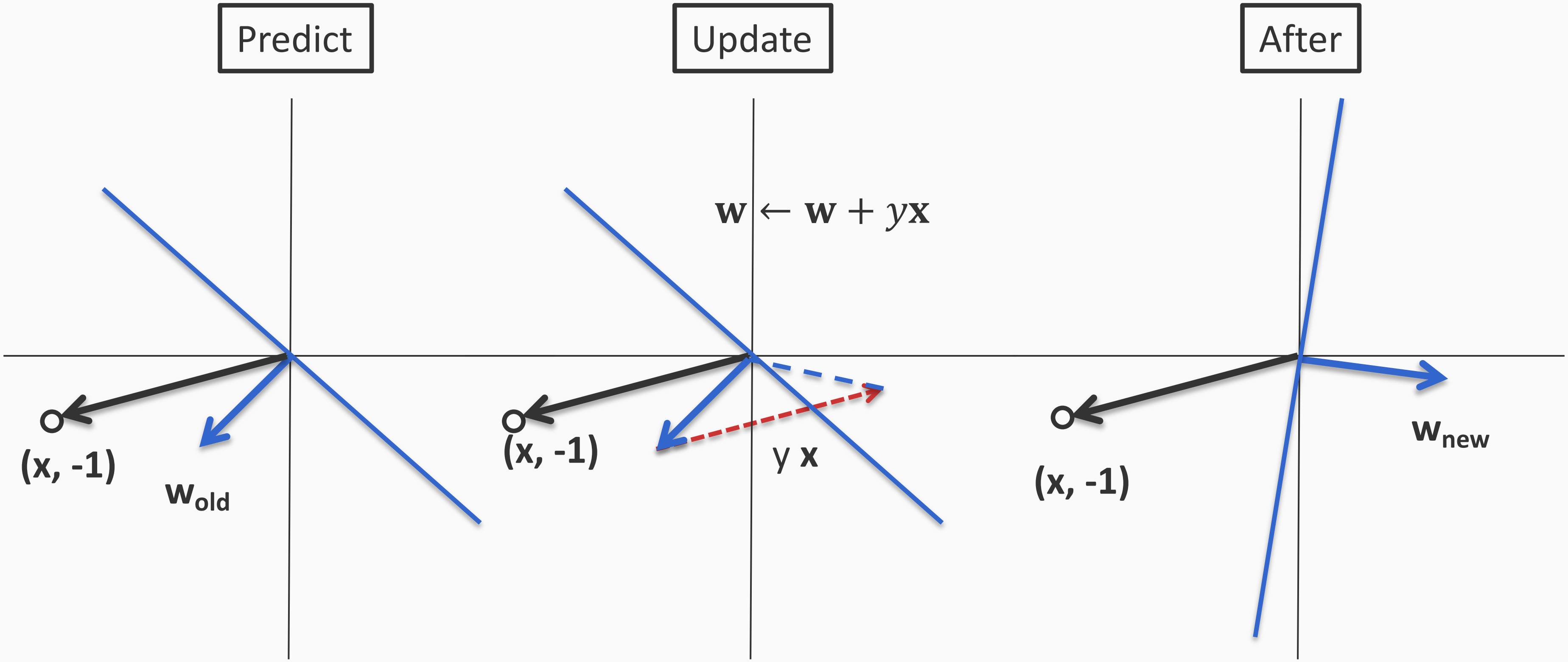

For a mistake on a negative example

Where are we?

- The Perceptron Algorithm
- Variants of Perceptron
- Perceptron Mistake Bound

Practical use of the Perceptron algorithm

1. Using the Perceptron algorithm with a finite dataset

2. Voting and Averaging

3. Margin Perceptron

1. The "standard" algorithm

Given a training set $D = \{(\mathbf{x}_i, y_i)\}$ where $\mathbf{x}_i \in \Re^n, y_i \in \{-1, 1\}$

1. Initialize $\mathbf{w} = \mathbf{0} \in \Re^n$
2. For epoch in $1 \cdots T$:
   1. Shuffle the data
   2. For each training example $(\mathbf{x}_i, y_i) \in D$:
      - If $y_i \mathbf{w}^T \mathbf{x}_i \leq 0$, then:
        – update $\mathbf{w} \leftarrow \mathbf{w} + r y_i \mathbf{x}_i$
3. Return $\mathbf{w}$

Prediction on a new example with features $\mathbf{x}$: $\text{sgn}(\mathbf{w}^T \mathbf{x})$

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T is a hyper-parameter to the algorithm

Another way of writing that there is an error

Prediction on a new example with features $\mathbf{x}$: $\mathrm{sgn}(\mathbf{w}^T \mathbf{x})$

2. Voting and Averaging

- So far: We return the final weight vector

- Voted perceptron
  - Remember every weight vector in your sequence of updates.
  - At final prediction time, each weight vector gets to vote on the label. The number of votes it gets is the number of iterations it survived before being updated
  - Comes with strong theoretical guarantees about generalization, impractical because of storage issues

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  - Comes with strong theoretical guarantees about generalization, impractical because of storage issues
- Averaged perceptron
  - Instead of using all weight vectors, use the average weight vector (i.e longer surviving weight vectors get more say)
  - More practical alternative and widely used

Averaged Perceptron

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This is the simplest version of the averaged perceptron

There are some easy programming tricks to make sure that **a** is also updated only when there is an error

Prediction on a new example with features $\mathbf{x}$: $\mathrm{sgn}(\mathbf{a}^T \mathbf{x})$

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If you want to use the Perceptron algorithm, use averaging

Prediction on a new example with features $\mathbf{x}$: $\mathrm{sgn}(\mathbf{a}^T \mathbf{x})$

3. Margin Perceptron

- Perceptron makes updates only when the prediction is incorrect

$$y_i \mathbf{w}^T \mathbf{x}_i \leq 0$$

- What if the prediction is close to being incorrect? That is, Pick a small positive η and update when

$$y_i \mathbf{w}^T \mathbf{x}_i \leq \eta$$

- Can generalize better, but need to choose η

  Exercise: Why is the margin perceptron a good idea?

The Perceptron

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The hype

NEW NAVY DEVICE LEARNS BY DOING

Psychologist Shows Embryo of Computer Designed to Read and Grow Wiser

WASHINGTON, July 7 (UPI) —The Navy revealed the embryo of an electronic computer today that it expects will be able to walk, talk, see, write, reproduce itself and be conscious of its existence.

The embryo—the Weather Bureau's $2,000,000 "704" computer—learned to differentiate between right and left after fifty attempts in the Navy's demonstration for newsmen.

The service said it would use this principle to build the first of its Perceptron thinking machines that will be able to read and write. It is expected to be finished in about a year at a cost of $100,000.

The New York Times, July 8 1958

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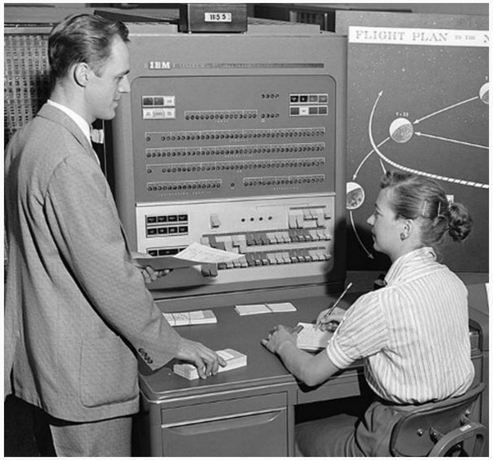

The IBM 704 computer

What you need to know

- The Perceptron algorithm
- The geometry of the update
- What can it represent
- Variants of the Perceptron algorithm